Improved stability bound for steady-state flow in a car-following model of road traffic on a circular route

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This note revisits a car-following model of road traffic on a circular route that was studied in recent literature, and improves a stability result for steady-state flows that was obtained in this literature. It will be shown through a counter example that the stability bound obtained in the literature only gives a sufficient condition for stability, which only becomes necessary when the number of cars on the route tends to infinity. We will further present a result that gives a necessary and sufficient condition for stability.

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I. INTRODUCTION

The objective of this paper is to revisit a car-following model of road traffic on a circular route that was studied in [1,3], and to improve a stability result that was obtained in these papers. Motivated by a recent paper by the author ([2]), it will be shown through a counter example that the stability bound obtained in [1,3] only gives a sufficient condition for stability, which only becomes necessary when the number of cars on the route tends to infinity. We will further present a result that gives a necessary and sufficient condition for stability.

We will first introduce the car-following model that was studied in Refs. [1], [3]. In this model, a circular road of length L is considered, on which N cars are present. It is assumed that each of the cars has a legal velocity function V, which depends on the distance to the car just ahead. It is further assumed that each driver responds to a stimulus from the vehicle ahead in such a way that he can maintain the legal safe velocity V. The function V typically has the properties that $V(0)=0$, and that it is bounded and strictly increasing. Denoting by x_j the position of the jth car ($j=1,...,N$), this then leads to a model of the form

$$\ddot{x}_j=-a[\dot{x}_j-V(x_{j+1}-x_j)] \quad (j=1,...,N;\ \ x_{N+1}:=x_1+L-x_N), \tag{1}$$

where $a>0$ is a constant representing the driver's sensitivity, which is assumed to be the same for all drivers.

The model (1) possesses a steady-state flow (or a synchronous motion in the terminology of Ref. [2]) in which all cars drive at the same velocity with a constant spacing between the cars. This steady-state flow is of the form

$$\overline{x_j}(t)=bj+ct \quad (j=1,...,N), \tag{2}$$

with

$$b=\frac{L}{N}, \quad c=V(b), \tag{3}$$

where b is the constant spacing between cars and c is the constant velocity of the cars.

Now define

$$\beta:=aV'(b) \tag{4}$$

and

$$p(z):=(z^2+az+\beta)^N-\beta^N. \tag{5}$$

One then has that the steady-state flow is locally exponentially stable if and only if all nonzero roots of $p(z)$ are in $\mathbb{C}^-:=\{z\in\mathbb{C}|\mathrm{Re}(z)<0\}$ (cf. [3,2]). It is then claimed in Refs. [1], [3] that this is the case if and only if $\beta<a^2/2$, which is equivalent to

$$V'(b)<\frac{a}{2}. \tag{6}$$

However, this condition is not necessary, as the following counter example shows.

Example 1.1. Consider the case where $N=4$, $a=2$, $V'(b)=1.5$. Clearly, we then have that Eq. (6) does not hold. However, we have that $\beta=3$ and

$$p(z)=(z^2+2z+3)^4-3^4. \tag{7}$$

Numerical calculation of the roots of $p(z)$ gives roots at 0, -2, $-1.8960\pm1.6741i$, $-0.1040\pm1.6741i$, $-1.0000\pm2.2361i$, which are clearly all in $\mathbb{C}^-$. This shows that Eq. (6) is not a necessary condition for local exponential stability of steady-state flows.

In the following section, we will present conditions that are indeed necessary and sufficient conditions for local exponential stability of steady-state flows.

II. AN IMPROVED STABILITY BOUND

Define the polynomial $r(s)$ by

$$r(s):=(s+\beta)^N-\beta^N. \tag{8}$$

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Note that we then have that $p(z)=r[z(z+a)]$. The following result is our starting point in deriving necessary and sufficient conditions for local exponential stability of steady-state flows.

Lemma 2.1. All nonzero roots of $p(z)$ are in $\mathbb{C}^-$ if and only if all nonzero roots s of $r(s)$ satisfy

$$\mathrm{Im}(s)^2 < -a^2\,\mathrm{Re}(s). \tag{9}$$

Proof. Since $p(z)=r[z(z+a)]$, we have that all nonzero roots of $p(z)$ are in $\mathbb{C}^-$ if and only if for all nonzero roots s of $r(s)$ we have that the solutions of $z(z+a)=s$ are in $\mathbb{C}^-$. The latter equality can be rewritten as

$$\left(z+\frac{a}{2}\right)^2 = s+\frac{a^2}{4} =: r\exp(i\theta), \tag{10}$$

which gives that

$$z=-\frac{a}{2}\pm\sqrt{r}\exp\left(\frac{i\theta}{2}\right).$$

This then gives that we should have that

$$0>-\frac{a}{2}+\sqrt{r}\cos(\tfrac{1}{2}\theta)=-\frac{a}{2}+\left[r\left(\frac{1+\cos\theta}{2}\right)\right]^{1/2},$$

which is equivalent to

$$r(1+\cos\theta)<\frac{a^2}{2}. \tag{11}$$

Write $s=x+iy$. Then it follows from Eq. (10) that $r=\sqrt{(x+a^2/4)+y^2}$ and $r\cos\theta = x+a^2/4$. Thus, Eq. (11) is equivalent to

$$\left[\left(x+\frac{a^2}{4}\right)+y^2\right]^{1/2}<\frac{a^2}{2}-\left(x+\frac{a^2}{4}\right). \tag{12}$$

Squaring both sides of this equality then gives that Eq. (12) is equivalent to

$$y^2<-a^2x, \tag{13}$$

which is exactly the inequality (9). This establishes our claim.

Using Lemma 2.1, we obtain the following result for local exponential stability of steady-state flows.

Theorem 2.2. Steady-state flows are locally exponentially stable if and only if

$$\left[1+\cos\left(\frac{2\pi}{N}\right)\right]V'(b)<a. \tag{14}$$

Proof. From Eq. (8) it follows that the nonzero roots of $r(s)$ are given by

$$s_k=-\beta+\beta\exp\left(\frac{2k\pi i}{N}\right)\quad(k=1,\ldots,N-1).$$

Now assume that Eq. (14) holds, and consider a nonzero root s_k $(k\in\{1,\ldots,N-1\})$ of $r(s)$. We then have that

$$\mathrm{Re}(s)=-\beta+\beta\cos\left(\frac{2k\pi}{N}\right),\quad \mathrm{Im}(s)=\beta\sin\left(\frac{2k\pi}{N}\right),$$

which gives that

$$\begin{aligned}\mathrm{Im}(s)^2+a^2\,\mathrm{Re}(s)&=\beta^2\sin^2\left(\frac{2k\pi}{N}\right)-a^2\beta\left[1-\cos\left(\frac{2k\pi}{N}\right)\right]\\&=\beta\left[1-\cos\left(\frac{2k\pi}{N}\right)\right]\left\{\beta\left[1+\cos\left(\frac{2k\pi}{N}\right)\right]\right.\\&\quad\left.-a^2\right\}.\end{aligned}$$

This gives that $\mathrm{Im}(s)^2+a^2\,\mathrm{Re}(s)<0$ if and only if

$$a^2>\beta\left[1+\cos\left(\frac{2k\pi}{N}\right)\right]\quad(k=1,\ldots,N-1). \tag{15}$$

Now assume that Eq. (15) holds for $k=1$. Then we have for $k=2,\ldots,N-1$:

$$\begin{aligned}\beta\left[1+\cos\left(\frac{2k\pi}{N}\right)\right]-a^2&<\beta\left\{\left[1+\cos\left(\frac{2k\pi}{N}\right)\right]\right.\\&\quad\left.-\left[1+\cos\left(\frac{2\pi}{N}\right)\right]\right\}=-\beta\left[\cos\left(\frac{2\pi}{N}\right)-\cos\left(\frac{2k\pi}{N}\right)\right]\\&=-2\beta\sin\left(\frac{(k+1)\pi}{N}\right)\sin\left(\frac{(k-1)\pi}{N}\right)\leqslant 0,\end{aligned}$$

where the last inequality follows from the fact that for $k=2,\ldots,N-1$ we have that

$$0<\frac{\pi}{N}\leqslant\frac{(k-1)\pi}{N},\frac{(k+1)\pi}{N}\leqslant\pi.$$

Thus, we have that Eq. (15) holds for all $k\in\{1,\ldots,N-1\}$ if and only if it holds for $k=1$. It is now straightforwardly checked that Eq. (15) for $k=1$ is equivalent to Eq. (14), which establishes our claim.

Remark 2.3. Note that for $N\to\infty$, the left hand side of Eq. (14) tends to $2V'(b)$. This shows that the condition for stability given in Refs. [1], [3] only is a necessary condition for $N\to\infty$. Note further that for $N=2$ the right hand side of Eq. (14) equals zero. Thus, for $N=2$ the steady-state flow is always locally exponentially stable.

III. CONCLUSIONS

This paper has given a generalized stability criterion for steady-state flows of road traffic on a circular route. The criterion generalizes the well-known result for an infinite number of vehicles presented in Refs. [1], [3] to finite numbers, and shows that finite-size effect tend to stabilize traffic.

It is to be noted that the circular route infrastructure is not very relevant for real (motorway) traffic. However, in real motorway traffic models it is often used as a test scenario. Moreover, two recent papers [2,4] have shown that the circular route infrastructure *is* of relevance when one considers traffic on bus routes.

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